

# UNIVERSAL GRAPHS WITHOUT INSTANCES OF CH: REVISITED

BY

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## ABSTRACT

We give a complete proof<sup>††</sup> of the consistency of the existence of a universal graph of power  $\lambda$ , where  $\kappa = \kappa^{<\kappa} < \lambda = \text{cf } \lambda < 2^\kappa$  are arbitrary.

## Introduction

The problem of the existence of universal models is very natural and appears in several contexts. Note that the related problem of the existence (for first order  $T$ ) of saturated models (which is more central for the model theorist, but not so adopted elsewhere) is completely resolved — for a first order  $T$ ,  $T$  has a saturated model in  $\lambda$  iff  $\lambda = \lambda^{<\lambda} \geq |D(T)|$  or  $T$  is stable in  $\lambda$  (see [6], VIII, §4 and reinforcing there). Also note that when a universal homogeneous, or saturated, or even just special model exists, a universal one exists (the same) (see works of Jonsson and of Morley and Vaught). So if  $T$  is first order,  $\lambda = \lambda^{<\lambda} > |T|$  or  $\lambda > |T|$  strong limit, then  $T$  has a universal model in  $\lambda$ . But, of course, this may be rare.

To get non-existence of a universal model in  $\lambda$  is not hard — if we add

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<sup>††</sup> The proof in the second section of [9] is flawed.

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$\aleph_2$ -Cohen reals any non- $\aleph_0$ -stable countable  $T$  has no universal model in  $\aleph_1$ ; if  $\lambda = \lambda^{<\lambda}$  and we add  $\mu$  Cohen subsets of  $\lambda$ , no unstable  $T$  has a universal model in any  $\chi \in (\lambda, \mu)$  (see representation in Mekler [5]).

In [7] we show that the theory of linear order may have a universal model in  $\aleph_1$  though  $\aleph_1 < 2^{\aleph_0}$  (using a combination of iteration of proper forcing and oracles). We also show even stronger results (categoricity of a PC class) for a natural unsuperstable theory.

In [9] we show the same for the theory of graphs (the method is related to the one in Abraham, Rubin and Shelah [1]). Here we generalize [9] to higher cardinals.

Meanwhile Mekler [5] generalizes [7] and [9] for a family of universal theories with strong amalgamation properties ( $\mathcal{P}^-(3)$ ,  $\mathcal{P}^-(4)$  respectively).

The author found examples of countable theories which never have a universal model in  $\aleph_1 < 2^{\aleph_0}$ , but are  $\aleph_0$ -categorical and with amalgamation.<sup>†</sup>

## §1.

1.1. THEOREM. *Suppose G.C.H. for simplicity,  $\kappa < \lambda \leq \mu$ ,  $\kappa$  and  $\lambda$  are regular and  $\mu^{<\lambda} = \mu$ . Then for some forcing notion  $P^*$ :*

- (1)  $P^*$  is  $\kappa$ -complete and has power  $\mu$ .
- (2)  $P^*$  does not collapse any  $\chi$ ,  $\kappa < \chi \leq \mu$ .
- (3)  $\Vdash_{P^*}$  "there is a universal graph of power  $\lambda$ ".
- (4)  $\Vdash_{P^*}$  " $2^\kappa = \mu$ ".

PROOF. By the proof of Baumgartner [2] Theorem 6.1 we easily get:

1.2. PRELIMINARY FORCING. For some forcing notion  $P$ ,  $|P| = \mu$ ,  $P$  is  $\kappa$ -complete and satisfies the  $(\lambda)^+$ -c.c., does not collapse cardinals and does not change cofinalities, and in  $V_a = V^P$  there is a family  $\mathcal{U} = \{A_\alpha : \alpha < \mu\}$  of  $\mu$  subsets of  $\lambda$  such that:

- (a)  $A \neq B \in \mathcal{U}$  implies  $|A \cap B| < \kappa$ .
- (b)  $A_\alpha$  is a stationary subset of  $\{\delta < \lambda : \text{cf } \delta = \kappa\}$ .

1.3. GENERAL DESCRIPTION. We shall define a  $(<\kappa)$ -support iteration of forcing notions satisfying the  $\kappa^+$ -c.c.  $\tilde{Q} = \langle P_\alpha, \tilde{Q}_\alpha : \alpha < \mu \rangle$ .  $Q_0$  forces a graph  $(\lambda, R_0)$ , which shall be a universal graph of power  $\lambda$ . We shall define the  $\tilde{Q}_\alpha$  by induction on  $\alpha$  (together with some auxiliary things), and will have to prove that it satisfies the  $\kappa^+$ -c.c.

<sup>†</sup> Added in proof. See much more in M. Kojman and S. Shelah, in preparation.

In stage  $\alpha > 0$  we will have a  $P_\alpha$ -name  $\underline{R}_\alpha$  so that  $\Vdash_{P_\alpha} "(\lambda, \underline{R}_\alpha) \text{ is a graph}"$  and in  $V_a^{P_\alpha}$ ,  $\tilde{Q}_\alpha$  will force an embedding  $f_\alpha$  of the graph  $(\lambda, \underline{R}_\alpha)$  into the graph  $(\lambda, \underline{R}_0)$ . It is known that we can take care that every  $P_\mu$ -name of a graph on  $\lambda$  appears as  $(\lambda, \underline{R}_\alpha)$  for some  $\alpha < \mu$ .

The problem is, of course, that the various  $f_\alpha$  may give contradicting demands on  $(\lambda, \underline{R}_0)$ . In order to avoid this as much as possible we shall make the  $f_\alpha$ 's such that for  $\beta < \alpha$  the set  $(\text{Rang } f_\alpha) \cap (\text{Rang } f_\beta)$  has cardinality  $< \kappa$ . It is reasonable to demand that " $\text{Rang } f_\alpha \subseteq A_\alpha$ ".

#### 1.4. THE FULL INDUCTIVE DEFINITION. We let

$$Q_0 = \{(w, r) : w \text{ a subset of } \lambda \text{ of power } < \kappa, \\ r \text{ a reflexive symmetric two-place relation on } w\}.$$

The order on  $Q_0$  is:  $q_1 \leq q_2$  iff  $q_1$  is a submodel of  $q_2$ . Let  $\underline{R}_0$  be the  $Q_0$ -name  $\bigcup \tilde{Q}_{Q_0}$  so  $\underline{R}_0$  is a two-place reflexive symmetric relation on  $\lambda$ .

Now  $F$  will be a function such that for each  $(< \kappa)$ -support iteration (see 1.6)  $\tilde{Q}^\gamma = \langle P_\alpha, \tilde{Q}_\beta : \beta < \gamma, \alpha \leq \gamma \rangle$ ,  $F(\tilde{Q}^\gamma)$  is a  $P_\gamma$ -name of a graph  $(\lambda, R)$ . Let  $\chi$  be a large enough regular cardinal.

Now for each  $\alpha > 0$ , we let  $(\lambda, \underline{R}_\alpha) = F(\langle P_\beta, \tilde{Q}_\beta : \beta < \alpha \rangle)$ , and we shall define  $\langle N_{\alpha,i} : i < \lambda \rangle$ , and  $\tilde{Q}_\alpha$ .

First let  $\langle N_{\alpha,i} : i < \lambda \rangle$  be a sequence of elementary submodels of  $(H(\chi)^{V_\alpha}, \in)$  such that for  $j < \lambda$ :

$$\langle P_\beta, \tilde{Q}_\beta : \beta < \alpha \rangle, (\lambda, \underline{R}_\alpha), \text{ and } \langle A_\beta : \beta < \mu \rangle \text{ belong to } N_{\alpha,j},$$

$\|N_{\alpha,j}\| < \lambda$ ,  $N_{\alpha,j} \cap \lambda$  is an ordinal and  $N_{\alpha,j}$  increasing continuous in  $j$ ;  $\langle N_{\alpha,i} : i \leq j \rangle \in N_{\alpha,j+1}$ . Note that this is done in  $V_a$ , so  $\langle N_{\alpha,i} : i < j \rangle \in V_a$  and even  $\langle \langle N_{\beta,i} : i < \lambda \rangle : \beta \leq \alpha \rangle \in V_a$ .

Define  $\xi_\alpha(i) = \sup(N_{\alpha,i} \cap \lambda)$ . Note that  $\xi_\alpha(i)$  is always a limit ordinal and  $\langle \xi_\alpha(i) : i < \lambda \rangle$  is increasing continuous. As  $\tilde{A}_\alpha$  is a stationary subset of  $\lambda$ , w.l.o.g.

$$(**) \quad \xi_\alpha(i) \in \tilde{A}_\alpha \quad \text{for every non-limit } i < \lambda.$$

We let  $A'_\alpha = \{\xi_\alpha(i+1) : i < \lambda\}$  and note that  $A'_\alpha \in V_a$ .

Now we come to the main point: defining  $\tilde{Q}_\alpha$  (in  $V_a^{P_\alpha}$ ):

(A) A member of  $\tilde{Q}_\alpha$  will consist of  $< \kappa$  many atomic conditions (see (B)) with no two of them explicitly contradictory (see (C)).

(B) There are two kinds of atomic conditions:

(I)  $f_\alpha(i) = j$  where  $i < j$ ,  $j \in A'_\alpha$  and  $j \in \{\xi_\alpha(\gamma+1) : \kappa i \leq \gamma < \kappa i + \kappa\}$  (or if you want, the sequence  $\langle \alpha, 0, i, j \rangle$ , is a condition).

(II)  $i \notin \text{Rang } \tilde{f}_\alpha$ .

(C) We shall have to say when two atomic conditions are explicitly contradictory; this occurs just in one of the following three cases:

(α) One-to-one:  $\tilde{f}_\alpha(i_1) = j_1$  and  $\tilde{f}_\alpha(i_2) = j_2$  when

$$i_1 = i_2, j_1 \neq j_2 \text{ or } i_1 \neq i_2, j_1 = j_2.$$

(β) Embedding:  $\tilde{f}_\alpha(i_1) = j_1$  and  $\tilde{f}_\alpha(i_2) = j_2$  when

$$V_{\alpha^*}^P \Vdash \text{"} i_1 R_\alpha i_2 \equiv \neg j_1 R_0 j_2 \text{"}.$$

(γ) Range:  $\tilde{f}_\alpha(i) = j$  and  $j \notin \text{Rang } \tilde{f}_\alpha$ .

The order is inclusion.

*Explanations.* The demand in (B)(I) is in order that  $Q_\alpha$  satisfies the  $\kappa^+$ -c.c. Each  $i < \lambda$  should have only  $\kappa$  many possible images. Why in (B)(I),  $j \in A'_\alpha$ ? For reasons similar to those in the club method (see [1]).

1.5. FACT. If  $P_\alpha$  satisfies the  $\kappa^+$ -c.c. then  $Q_\alpha$  gives an embedding.

We want to prove (in  $V_{\alpha^*}^P$ ) that  $\Vdash_{Q_\alpha} \text{"} (\lambda, R_\alpha) \text{ is embeddable into } (\lambda, R_0) \text{"}$ . We have a natural name for exemplifying this:  $\tilde{f}_\alpha$  (defined by  $\tilde{f}_\alpha(i) = j$  iff  $\Vdash_{Q_\alpha} [\tilde{f}_\alpha(i) = j]$  belongs to the generic subset of  $Q_\alpha$ ). It is (forced to be) a partial function from  $\lambda$  to  $\lambda$  by 1.4(B)(I), one-to-one by 1.4(C)(α) and an embedding by 1.4(C)(β). But we should still prove that for every  $i < \lambda$ ,  $\Vdash_{Q_\alpha} \text{"} i \in \text{Dom } \tilde{f}_\alpha \text{"}$ . This is equivalent to proving that for every  $q \in Q_\alpha$  for some  $j$ ,  $q \cup \{[\tilde{f}_\alpha(i) = j]\} \in Q_\alpha$  (assuming  $q$  itself has no such member). By 1.4(B)(I) we have  $\kappa$  many candidates for  $j$ :

$$B = \{ \xi_\alpha(j+1) : \kappa i < j+1 < \kappa i + \kappa \}.$$

The only difficult demand comes from 1.4(C)(β). As  $B \in V_\alpha$  (as  $\langle N_{\alpha,i} : i < \lambda \rangle \in V_\alpha$ ), and as forcing by  $P_\alpha$  adds no new subset of  $\lambda$  of cardinality  $< \kappa$ , by the definition of  $Q_0$ ,  $\kappa$  many  $j \in B$  satisfy this, so we finish the proof of 1.5.

Now the rest of the proof is dedicated to proving that  $P_\alpha$  satisfies the  $\kappa^+$ -c.c. assuming this holds for all  $\beta < \alpha$ . For this we shall derive more detailed information on  $\tilde{Q}^\alpha$  (using the fact that all  $\tilde{Q}_\beta$ ,  $\beta < \alpha$ , were defined as above).

1.6. NICE DENSE SUBSETS OF  $P_\alpha$ . For a function  $p$  with  $\text{dom}(p) \subseteq \alpha$  and such that for all  $\alpha \in \text{dom}(p)$ ,  $p(\alpha) \in \tilde{Q}_\kappa$ , define  $\text{Dom}(p) = \{ \alpha \in \text{dom } p : p(\alpha) \neq 0 \}$ . We use the variant of  $(< \kappa)$ -support iteration in which

$$P_\beta = \{ p : p \text{ a function with domain } \subseteq \beta, \text{Dom}(p) \text{ is of power } < \kappa \\ \text{and } p \restriction \gamma \Vdash_{P_\gamma} \text{"} p(\gamma) \text{ is a member of } \tilde{Q}_\gamma \text{ and is set} \\ \text{of atomic condition of } \tilde{Q}_\gamma \text{"}, \text{ for } \gamma \in \text{Dom } p \},$$

Let

$$D_\beta^0 = \{ p \in P_\beta : \text{for each } \gamma \in \text{Dom } p, p(\gamma) \text{ is an actual} \\ \text{set of atomic conditions of power } < \kappa \}.$$

Note that not every function  $p$  with domain a subset of  $\beta$ , of power  $< \kappa$ , each  $p(\gamma)$  a set of  $< \kappa$  atomic conditions of the forms mentioned in 1.4(B), is in  $D_\beta^0$ : we need  $p \upharpoonright \gamma \Vdash_p "p(\gamma) \in Q_\gamma"$  for each  $\gamma \in \text{dom } p$ .

1.7. DEFINITION.  $D_\beta^1 = \{ p : p \text{ is a function with domain } \subseteq \beta \text{ of power } < \kappa \text{ and for } \gamma \in \text{Dom } p, p(\gamma) \text{ satisfies the demands for } p(\gamma) \in Q_\gamma \text{ in 1.4(A), (B), (C) except possibly "there are no two atomic conditions in } p(\gamma) \text{ which are explicitly contradictory by 1.4(C)(\beta)" } \}$ .

For  $p \in D_\beta^1$ ,  $\gamma \notin \text{Dom } p$ , let  $p(\gamma) = \emptyset$ .

We define an order on  $D_\beta^1$ :

$$p \leq r \text{ iff for every } \gamma, p(\gamma) \subseteq r(\gamma).$$

1.8. FACT. (1)  $D_\beta^0$  is a dense subset of  $P_\beta$  and  $D_\beta^0 \subseteq D_\beta^1$ .

(2) On  $D_\beta^1 \cap P_\beta$  the orders of  $P_\beta$  and of  $D_\beta^1$  coincide.

(3) For  $p \in D_\beta^1$ ,  $p \in D_\beta^0$  iff for every  $\gamma \in \text{Dom } p$  and

$$[f_\gamma(i_1) = j_1], [f_\gamma(i_2) = j_2] \text{ in } p(\gamma)$$

$$p \upharpoonright \gamma \Vdash_p "i_1 R_\gamma i_2 \text{ iff } j_1 R_\gamma j_2"$$

(prove  $p \upharpoonright \gamma \in P_\gamma$  by induction).

(4) If  $p \in D_\beta^1$ ,  $w \subseteq \text{Dom } p$  then  $p \upharpoonright w \in D_\beta^1$ .

1.9. FACT.

(1) If  $(\beta \leq \alpha \text{ and}) p_\zeta \in P_\beta$  for  $\zeta < \delta < \kappa$ , and  $p_\zeta \leq p_\xi$  for  $\zeta < \xi < \delta$  and  $p$  is defined by:  $\text{Dom}(p) = \bigcup_{\zeta < \delta} \text{Dom}(p_\zeta)$  and for  $\gamma \in \text{Dom}(p)$ ,  $p(\gamma) = \bigcup_{\zeta < \delta} p_\zeta(\gamma)$  (remember  $Q_\gamma$  is ordered by inclusion, and  $p_\zeta(\gamma) = \emptyset$  for  $\gamma \notin \text{Dom}(p_\zeta)$ ) then  $p \in P_\beta$  and  $p_\zeta \leq p$  for  $\zeta < \delta$  (remember the beginning of 1.6). We say in such cases  $p = \bigcup_{\zeta < \delta} p_\zeta$ .

(2) If  $p_\zeta \in D_\beta^1$  for  $\zeta < \delta < \kappa$ ,  $p_\zeta \leq p_\xi$  for  $\zeta \leq \xi < \delta$  and let  $p = \bigcup_{\zeta < \delta} p_\zeta$  be defined by  $\text{Dom}(p) = \bigcup_{\zeta < \delta} \text{Dom}(p_\zeta)$ ,  $p(\beta) = \bigcup_{\zeta < \delta} p_\zeta(\beta)$ , then

$$p \in D_\beta^1 \text{ and } p_\zeta \leq p \text{ for } \zeta < \delta.$$

(3) In (2), if  $p_\zeta \in D_\beta^0$  for  $\zeta < \delta$ , then

$$p \in D_\beta^0.$$

(4) If  $p^1, p^2 \in D_\beta^0$ , and for every  $\gamma \in \text{Dom } p^1 \cap \text{Dom } p^2$ ,  $p^1(\gamma) \subseteq p^2(\gamma)$  or

$p^2(\gamma) \subseteq p^1(\gamma)$ , then  $p^1 \cup p^2 \in D_\beta^0$  where  $(p^1 \cup p^2)(\gamma) = p^1(\gamma) \cup p^2(\gamma)$  for  $\gamma \in \text{Dom } p^1 \cup \text{Dom } p^2$  and  $p^1 \leq (p^1 \cup p^2)$ ,  $p^2 \leq (p^1 \cup p^2)$ .

Now we continue with

1.10. DEFINITION. For  $\gamma \leq \alpha$ ,  $q \in Q_\gamma$ , and ordinal  $\delta < \lambda$  (usually but not always limit) we let:

if  $\gamma > 0$ :  $q^{[\delta]} = \{[f_\gamma(i) = j] : [f_\gamma(i) = j] \in q$   
and for some  $\xi < \lambda$ ,  
 $j < \xi_\gamma(\xi) < \delta\}$   
 $\cup \{[j \notin \text{Rang } f_\gamma] : [j \notin \text{Rang } f_\gamma] \in q$   
and for some  $\xi < \lambda$ ,  $j < \xi_\gamma(\xi) < \delta\}$ ;  
 $q^{(\delta)} = \{[j \notin \text{Rang } f_\gamma] : [j \notin \text{Rang } f_\gamma] \in q\} \cup q^{[\delta]}$   
if  $\gamma = 0$ :  $q^{[\delta]} = q^{(\delta)} = q \upharpoonright \delta$ , i.e., if  $q = (w, \gamma)$ , then  
 $q^{[\delta]} = q^{(\delta)} = (w \cap \delta, r \upharpoonright (w \cap \delta))$ .

1.11. DEFINITION. (1) For  $p \in P_\alpha$ , and ordinal  $\delta < \lambda$ , let  $p^{[\delta]}$  be a function with domain  $\text{Dom } p$  and  $p^{[\delta]}(\gamma) = (p(\gamma))^{[\delta]}$ .

(2) We can make those definitions even for  $p \in D_\beta^1$ .

1.12. FACT. (1) For any ordinals  $\gamma > 0$ ,  $\delta$  and  $q \in Q_\gamma$ ,  $q^{[\delta]} = \emptyset$  or for some  $\xi$ ,  $q^{[\delta]} = q^{[\xi_\gamma(\xi)]}$ ,  $\xi_\gamma(\xi) \leq \delta$  and  $\xi$  limit or  $q^{[\delta]} = q^{[\xi_\gamma(\xi)+1]} = q^{[\xi_\gamma(\xi+1)]}$ ,  $\xi_\gamma(\xi) < \delta$ ,  $\xi$  successor.

- (2) If  $p \in D_{\beta+1}^1$ ,  $\xi \leq \lambda$  and  
 $(\forall a)[a \subseteq N_{\beta,\xi} \wedge |a| < \kappa \Rightarrow a \in N_{\beta,\xi}]$ , then  
 $(p^{[\xi_\beta(\xi)]} \upharpoonright |N_{\beta,\xi}|) \in N_{\beta,\xi}$ .  
(3) If  $p \in D_\beta^1$ ,  $\delta < \lambda$ , then  $p^{[\delta]} \in D_\beta^1$  and  $p^{(\delta)} \in D_\beta^1$ .  
(4)  $p^{[\delta]} \leq p^{(\delta)} \leq p$  (in  $D_\beta^1$ ).  
(5) If  $p \in D_\beta^0$ ,  $p \leq r \in D_\beta^1$ ,  $r = r^{(\delta)}$ ,  $r^{[\delta]} \leq p$ , then  $r \in D_\beta^0$ .  
(6) If  $\delta$  is limit ordinal then for  $p \in D_\beta^1$ ,  
 $p^{[\delta]} = \cup \{p^{[\alpha]} : \alpha < \delta\}$ .  
(7) If  $\delta_1 \leq \delta_2$ ,  $p \in D_\beta^1$  then  $p^{[\delta_1]} \leq p^{[\delta_2]}$ .

1.13. DEFINITION. Let

$D_\gamma = \{p \in D_\gamma^0 : \text{for every } \delta < \lambda, p^{[\delta]} \in D_\gamma^0;$   
moreover, if  $0 < \beta \in \text{Dom } p$  and for  $l = 1, 2$ ,  
 $[f_\beta(i_l) = j_l] \in p(\beta)$ ,  $i_1, i_2 < \xi_\beta(\xi)$  then  
 $p^{[\xi_\beta(\xi)]} \upharpoonright \beta \Vdash_{P_\beta} "i_1 R_\beta i_2"$  or  $p^{[\xi_\beta(\xi)]} \upharpoonright \beta \Vdash_{P_\beta} "\neg i_1 R_\beta i_2"$  }.

1.13A. FACT. The second condition ("Moreover ...") in the definition of  $D_\gamma$  implies the first.

PROOF. Prove by induction on  $\beta \leq \gamma$  that for every  $\delta$ ,  $(p \upharpoonright \beta)^{[\delta]} \in D_\beta$ .

1.14. THE CRUCIAL CLAIM.  $D_\beta$  is a dense subset of  $P_\beta$  (for  $\beta \leq \alpha$ ).

PROOF. We prove this by induction on  $\beta$ .

Case i.  $\beta = 0$ .

Nothing to prove.

Case ii.  $\beta = 1$ .

Clearly (as  $Q_0$  is so simple).

Case iii.  $\beta$  limit of cofinality  $\geq \kappa$ .

Trivial, as  $P_\beta = \bigcup_{\gamma < \beta} P_\gamma$ .

Case iv.  $\beta$  limit of cofinality  $< \kappa$ .

Let  $p \in P_\beta$  and we shall find  $q \in D_\beta$ ,  $p \leq q$ .

Let  $\beta = \bigcup_{\zeta < \text{cf } \beta} \beta(\zeta)$ ,  $\beta(\zeta)$  increasing continuous and let  $\beta(\text{cf } \beta) \stackrel{\text{def}}{=} \beta$ . We define by induction on  $\zeta \leq \text{cf } \beta$  a condition  $q_\zeta \in P_{\beta(\zeta)}$ ,  $q_\zeta \in D_{\beta(\zeta)}$ ,  $q_\zeta$  increasing and  $p \upharpoonright \beta(\zeta) \leq q_\zeta$ . For  $\zeta = 0$  use the induction hypothesis on  $\beta(0)$  (and  $p \upharpoonright \beta(0)$ ). For limit,  $q_\zeta \stackrel{\text{def}}{=} \bigcup_{\xi < \zeta} q_\xi$  is in  $D_{\beta(\zeta)}$  (as for each  $\delta$ ,  $q_\zeta^{[\delta]} = \bigcup_{\xi < \zeta} q_\xi^{[\delta]}$  is in  $D_{\beta(\zeta)}^0$  by Fact 1.9(3)), and clearly it is  $\geq p \upharpoonright \beta(\zeta)$  (as  $\langle \beta(\xi) : \xi \leq \text{cf}(\beta) \rangle$  is increasing continuous). For successor  $\zeta$ , use the induction hypothesis and 1.9(4). So  $q_\zeta$  is as required. This applies in particular to  $\zeta = \text{cf } \beta$ .

Case v.  $\beta = \gamma + 1$ ,  $\gamma > 0$ .

So suppose  $p \in P_\beta$  and we shall find  $p^1 \geq p$ ,  $p^1 \in D_\beta$ . First, by Fact 1.8(1) there is  $p_1 \geq p$ ,  $p_1 \in D_\beta^0$ . Second, by the induction hypothesis there is  $r \in D_\gamma$ ,  $r \geq p_1 \upharpoonright \gamma$ . As  $p_1 \in D_\beta^0$  by 1.12(1) there is an increasing continuous sequence  $\langle \delta(\theta) : \theta \leq \theta(*) \rangle$  of ordinals  $< \lambda$ ,  $\delta(0) = 0$ ,  $\theta(*) < \kappa$ ,  $\delta(\theta + 1) \in \{ \xi_\gamma(\xi + 1) : \xi < \lambda \}$  such that, if  $[f_\gamma(i) = j] \in p_1(\gamma)$ , then  $\text{Min}\{ \xi : \xi_\gamma(\xi) > i \}$  is  $\delta(\theta + 1)$  for some  $\theta$  ( $\varepsilon$  is necessarily non-limit).

We now define by induction on  $\theta \leq \theta(*)$  a condition  $r_\theta$  such that:

- (\*) (i)  $r_\theta \in D_\gamma$ ,
- (ii)  $r_0 \geq p_1 \upharpoonright \gamma$ ,
- (iii)  $r_\theta$  is increasing continuous,
- (iv) if  $[f_\gamma(i_1) = j_1]$ ,  $[f_\gamma(i_2) = j_2]$  belongs to  $p(\gamma)$  and  $i_1, i_2 < \delta(\theta)$  then  $r_\theta^{[\delta(\theta)]}$  determines the truth value of  $i_1 R_\gamma i_2$ .

If we succeed we shall finish to prove the crucial claim:  $p_1 \cup r_{\theta(*)}$  is as required: for each  $\delta$  choose  $\theta$  such that  $\delta \geq \delta(\theta)$ ,  $p_1(\gamma)^{[\delta]} = p_1(\gamma)^{[\delta(\theta)]}$  so by (\*) above

$$r_{\theta(*)}^{[\delta(\theta)]} \Vdash_{P_\gamma} "p_1(\gamma)^{[\delta(\theta)]} \in \tilde{Q}_\gamma"$$

but  $r_{\theta(*)}^{[\delta(\theta)]} \leq r_{\theta(*)}^{[\delta]}$  hence

$$r_{\theta(*)}^{[\delta]} \Vdash_{P_\gamma} "p_1(\gamma)^{[\delta]} = p_1(\gamma)^{[\delta(\theta)]} \in \tilde{Q}_\gamma"$$

but  $r_{\theta(*)} \in D_\gamma$  hence  $r_{\theta(*)}^{[\delta]} \in D_\gamma^0$  hence  $r_{\theta(*)}^{[\delta]} \cup p_1^{[\delta]} \in D_{\gamma+1}^0$  as required.

So for proving the crucial claim we just have to carry the induction definition of  $r_\theta$  as to satisfy (\*). For  $\theta = 0$ ,  $r_\theta = r$  (remember  $p_1(\gamma)^{[0]} = \emptyset$ ), for  $\theta$  limit  $r_\theta = \bigcup_{\sigma < \theta} r_\sigma$ , and there are no problems (see 1.9). So let  $\theta > 0$  be a successor ordinal.

Let  $\xi$  be such that  $\xi_\gamma(\xi) = \delta(\theta)$  (exists as  $\theta$  is a successor ordinal). So clearly  $p_1(\gamma)^{[\delta(\theta)]} \subseteq N_{\gamma, \xi}$  (and if  $(\forall \chi < \lambda) \chi^{<\lambda} < \lambda$  then w.l.o.g.  $p_1(\gamma)^{[\delta(\theta)]} \in N_{\gamma, \xi}$ ; otherwise this is not necessarily true). But for every finite subset  $u$  of  $p_1(\gamma)^{[\delta(\theta)]}$ ,  $u \in N_{\gamma, \xi}$  and let

$$I_u = \{r \in P_\gamma : r \in D_\gamma, \text{ and either } r \Vdash_P "u \text{ satisfies 1.4(C)(}\beta\text{)}" \\ \text{or } r \Vdash_P "u \text{ fails 1.4(C)(}\beta\text{)}"\}$$

and let  $\{u_{\theta, \sigma} : \sigma < \sigma_\theta\}$  list the finite subsets of  $p_1(\gamma)^{[\delta(\theta)]}$ . Clearly for each  $u_{\theta, \sigma}$ ,  $I_{u_{\theta, \sigma}}$  belongs to  $N_{\gamma, \xi}$  and it is a dense subset of  $P_\gamma$ . As  $P_\gamma$  satisfies the  $\kappa^+$ -c.c., necessarily  $N_{\gamma, \xi} \cap I_{u_{\theta, \sigma}}$  is a predense subset of  $P_\gamma$ . So we can define by induction on  $\sigma \leq \sigma_\theta$ ,  $r_{\theta, \sigma} \in D_\gamma$ ,  $r_{\theta, \sigma}$  increasing, and  $q_{\theta, \sigma} \in I_{u_{\theta, \sigma}} \cap N_{\gamma, \xi}$ , such that  $q_{\theta, \sigma} \leq r_{\theta, \sigma}$ .

Now  $q_{\theta, \sigma} \Vdash_P "u_{\theta, \sigma} \text{ fail 1.4(C)(}\beta\text{)}"$  is impossible as  $q_{\theta, \sigma}$  is compatible with  $r_\theta$  ( $r_{\theta, \sigma+1}$  exemplifies this) hence with  $r_0 \geq p_1 \upharpoonright \gamma$ , but  $p_1 \upharpoonright \gamma \Vdash_{P_\gamma} "p_1(\gamma) \in \tilde{Q}_\gamma"$  hence  $p_1 \upharpoonright \gamma \Vdash_{P_\gamma} "u_{\theta, \sigma} \in \tilde{Q}_\gamma"$ .

So  $q_{\theta, \sigma} \Vdash_{P_\gamma} "u_{\theta, \sigma} \text{ satisfies 1.4(C)(}\beta\text{)}"$  hence  $r_{\theta, \sigma} \Vdash_{P_\gamma} "u_{\theta, \sigma} \text{ satisfies 1.4(C)(}\beta\text{)}"$  hence  $r_{\theta, \sigma_\theta} \Vdash_{P_\gamma} "every \text{ finite } u \subseteq p_1(\gamma)^{[\delta(\theta)]} \text{ satisfies 1.4(C)(}\beta\text{)}"$  so  $r_{\theta, \sigma_\theta} \Vdash_{P_\gamma} "p_1(\gamma)^{[\delta(\theta)]} \in \tilde{Q}_\gamma"$ . But for every  $\sigma < \sigma_\theta$ ,  $q_{\theta, \sigma} \leq (r_{\theta, \sigma})^{[\delta(\theta)]} \leq (r_{\theta, \sigma_\theta})^{[\delta(\theta)]}$ , hence  $(r_{\theta, \sigma_\theta})^{[\delta(\theta)]} \Vdash_{P_\gamma} "p_1(\gamma)^{[\delta(\theta)]} \in \tilde{Q}_\gamma"$ . So let  $r_{\theta+1} \stackrel{\text{def}}{=} r_{\theta, \sigma_\theta}$ , and it is as required, so we have proved 1.14.

1.15. FACT. Suppose  $\delta = \xi_\beta(\xi)$  and  $p \in P_\alpha$ ,  $\beta < \alpha$ ,  $p = p^{[\delta]}$ .

- (1) If  $\underline{z}$  is a  $P_\beta$ -name of an ordinal,  $\underline{z} \in N_{\beta, \xi}$  then for some  $q$ ,  $p \leq q \in P_\alpha$ ,  $q = q^{[\delta]}$  and  $q$  force a value for  $\underline{z}$ , and  $p \upharpoonright [\beta, \alpha) = q \upharpoonright [\beta, \alpha)$ .
- (2) We can do this simultaneously to  $< \kappa$  such names.
- (3) If  $u \subseteq \delta$ ,  $|u| < \kappa$ , then there is  $q$ ,  $p \leq q \in P_\alpha$ ,  $q = q^{[\delta]}$  and  $q$  forces a value to  $\dot{R}_\beta \upharpoonright u$ .



PROOF. (1) As is the definition of  $q_{\theta, \sigma+1}$  in the proof of 1.14.

(2) By 1.15(1) and 1.9.

(3) By 1.15(2).

1.16. MAIN LEMMA.  $P_\alpha$  satisfies the  $\kappa^+$ -c.c.

PROOF. Let  $p_\zeta \in P_\alpha$  for  $\zeta < \kappa^+$ , and for  $\zeta \neq \xi$  the conditions  $p_\zeta, p_\xi$  are not compatible, and we shall eventually derive a contradiction. Clearly we can replace  $\langle p_\zeta : \zeta < \kappa^+ \rangle$  by  $\langle p'_\zeta : \zeta < \kappa^+ \rangle$  if  $p'_\zeta \geq p_\zeta$ , and by  $\langle p_\zeta : \zeta \in A \rangle$  if  $A \subseteq \kappa^+$ ,  $|A| = \kappa^+$ . We shall use this freely.

W.l.o.g. for every  $\zeta$ :

- (a)  $p_\zeta \in D_\alpha$ .
- (b)  $0 \in \text{Dom } p_\zeta$ .
- (c) If  $\beta \neq \gamma \in \text{Dom } p_\zeta, j \in A'_\beta \cap A'_\gamma$ , then  $j$  belongs to the universe of  $p_\zeta(0)$ .
- (d) If  $[j \notin \text{Rang } f_\beta] \in p_\zeta(\beta)$  or  $[f_\beta(i) = j] \in p_\zeta(\beta)$  for some  $\beta$  and  $i$ , then  $j$  belongs to the universe of  $p_\zeta(0)$ .
- (e) If  $[f_\beta(i) = j] \in p_\zeta(\beta)$  and  $j_1 \in A'_\beta, \kappa i < j_1 < j$  then  $j_1$  belongs to the universe of  $p_\zeta(0)$ .
- (f) If  $j$  belongs to the universe of  $p_\zeta(0)$  and  $\beta \in \text{Dom}(p_\zeta)$  then  $[j \notin \text{Rang } f_\beta] \in p_\zeta(\beta)$  or  $(\exists i)[f_\beta(i) = j] \in p_\zeta(\beta)$ .

We can easily find  $\zeta < \xi < \kappa^+$  such that:

if  $\beta \in \text{Dom}(p_\zeta) \cap \text{Dom}(p_\xi)$  then  $p_\zeta(\beta) \cup p_\xi(\beta)$  belongs to  $D_\alpha^1$ .

Let  $w = \{\delta(\theta) : \theta < \theta(*)\}$  where  $\theta(*) < \kappa$  be such that:

- (1)  $\delta(\theta)$  is increasing continuous.
- (2)  $\delta(0) = 0, \delta(\theta) < \lambda$ .
- (3)  $\text{Dom}(p_\zeta(0)) \cup \text{dom}(p_\xi(0)) \subseteq \{\delta(\theta) : \theta < \theta(*)\}$ .
- (4) If  $\delta(\theta)$  is limit then  $\delta(\theta + 1) = \delta(\theta) + 1$ .

Note.  $(\forall \theta)[\theta \text{ limit } \theta \leq \theta(*) \rightarrow \text{cf}(\delta(\theta)) = \text{cf } \theta]$ .

Now we shall define  $r_\theta$  by induction on  $\theta \leq \theta(*)$  such that:

- (A)  $r_\theta \in D_\alpha^0, r_\theta$  increasing continuous (in  $\theta$ ).
- (B)  $r_\theta = r_\theta^{[\delta(\theta)]}$ .
- (C)  $p_\zeta^{[\delta(\theta)]} \leq r_\theta, p_\xi^{[\delta(\theta)]} \leq r_\theta$ .
- (D)  $p^{[\delta(\theta)]} \subseteq r_\theta$  where  
 $p = p_a \cup p_b$  where

$$p_a(\beta) = \{[j \notin \text{Rang } f_\beta] : \beta \in \text{Dom } p_\zeta \cup \text{Dom } p_\xi \text{ and} \\ \neg (\exists i)([f_\beta(i) = j] \in p_\zeta(\beta) \cup p_\xi(\beta)) \\ \text{and } j \in \text{dom}(p_\zeta(0)) \cup \text{dom}(p_\xi(0))\},$$

$$p_b(\beta) = \{[j \notin \text{Rang } f_\beta] : \beta \in \text{Dom } p_\zeta \cup \text{Dom } p_\xi \text{ and} \\ \neg (\exists i)([f_\beta(i) = j] \in p_\zeta(\beta) \cup p_\xi(\beta)) \\ \text{and for some } \gamma \in (\text{Dom } p_\zeta \cup \text{Dom } p_\xi), \gamma \neq \beta \text{ and } j \in A'_\beta \cap A'_\gamma\}.$$

Case I.  $\theta = 0$ .

Trivial.

Case II.  $\theta = 1$ .

Use 1.9(4) for  $p_\zeta^{[\delta(1)]}$ ,  $p_\xi^{[\delta(1)]}$ .

Case III.  $\theta$  limit.

So  $\delta(\theta) = \bigcup_{\sigma < \theta} \delta(\sigma)$ , and  $\bigcup_{\sigma < \theta} r_\sigma$  is as required.

Case IV.  $\theta = \sigma + 1$ ,  $\delta(\sigma)$  non-limit  $> 0$ .

Trivial.

Case V.  $\theta = \sigma + 1$ ,  $\sigma > 0$ , not Case IV.

Let  $u_\zeta = \{\beta < \alpha : \beta \in \text{Dom}(p_\zeta), \text{ and } (\exists i)([f_\beta(i) = \delta(\sigma)] \in p_\zeta(\beta))\}$  so  $u_\zeta$  has cardinality  $< \kappa$  and, for  $\beta \in u_\zeta$ , let  $i = i_{\zeta, \beta}$  be such that  $[f_\beta(i_{\zeta, \beta}) = \delta(\sigma)] \in p_\zeta(\beta)$ ; similarly for  $\xi$ .

1.16A. FACT. There is  $q_\sigma$  such that

- (1)  $r_\sigma \leq q_\sigma \in D_\alpha$ ,
- (2)  $q_\sigma^{[\delta(\sigma)]} = q_\sigma$ ,
- (3) for  $\beta \in u_\zeta$ ,  $q_\sigma \restriction \beta$  forces a value for  $\underline{R}_\beta \restriction (\{i : (\exists j)([f_\beta(i) = j] \in q_\sigma)\} \cup \{i_{\zeta, \beta}\})$ ,
- (4) for  $\beta \in u_\xi$ ,  $q_\sigma \restriction \beta$  forces a value for  $\underline{R}_\beta \restriction (\{i : (\exists j)([f_\beta(i) = j] \in q_\sigma)\} \cup \{i_{\xi, \beta}\})$ .

PROOF. By 1.15 and closure under union.

We now want to define  $r_\theta$ . Let  $r_\theta$  be defined as follows:

for  $\beta < \alpha$ ,  $\beta \neq 0$ ,

$$r_\theta(\beta) = q_\sigma(\beta) \cup p_\zeta^{[\delta(\theta)]}(\beta) \cup p_\xi^{[\delta(\theta)]}(\beta) \cup p^{[\delta(\theta)]}(\beta);$$

for  $\beta = 0$ ,

$r_\theta(0)$  has universe  $\text{dom}(q_\sigma(0)) \cup \{\delta(\sigma)\}$ , extends  $q_\sigma(0)$ , and  $p_\zeta^{[\delta(\theta)]}(0)$ ,  $p_\xi^{[\delta(\theta)]}(0)$  and:

(\*) suppose  $\beta \in \text{Dom}(p_\zeta)$ ,  $[f_\beta(i) = j] \in q_\sigma$  and  $[f_\beta(i_{\zeta,\beta}) = \delta(\sigma)] \in p_\zeta(\beta)$  then  
 $q_\sigma \uparrow \beta \Vdash iR_\beta i_{\zeta,\beta}$  iff  $q_0 \uparrow \beta \Vdash \neg iR_\beta i_{\zeta,\beta}$  iff  $r_\theta(0) \models jR_0\delta(\sigma)$

and

(\*\*) similarly for  $\xi$ .

Note that  $q_\sigma \uparrow \beta \Vdash iR_\beta i_{\zeta,\beta}$  iff  $q_\sigma \uparrow \beta \Vdash \neg iR_\beta i_{\zeta,\beta}$  by 16A. We should verify that  $r_\theta$  is as required.

*Point (i).* Why is  $r_\theta(0)$  well defined?

*A priori* we may have two conflicting demands on the truth value of  $j_1 R_0 j_2$ . We have five sources of such demands:  $q_\sigma p_\zeta^{[\delta(\theta)]}(0)$ ,  $p_\xi^{[\delta(\theta)]}(0)$ , (\*) and (\*\*).

The first and second do not contradict as  $q_\sigma = q_\sigma^{[\delta(\sigma)]}$  whereas  $(p_\zeta^{[\delta(\theta)]})^{[\delta(\sigma)]} = p_\zeta^{[\delta(\sigma)]} \leq r_\sigma \leq q_\sigma$ . Similarly the first and third do not contradict.

The second and third do not contradict by the choice of  $\zeta < \xi$ , i.e., as  $p_\zeta \cup p_\xi \in D_\alpha^1$  so  $p_\zeta(0) \cup p_\xi(0) \in Q_0$ .

Next, the first and fourth do not contradict as  $q_\sigma^{[\delta(\sigma)]} = q_\sigma$ , so for every  $j: jR_0\delta(\sigma) \notin q_\sigma(0)$  and  $\neg jR_0\delta(\sigma) \notin q_\sigma(0)$ . Similarly, the first and fifth do not contradict.

What about a contradiction between the second and fourth, i.e.,  $p_\zeta^{[\delta(\theta)]}(0)$  and an instance of (\*)? Let the instance of (\*) be  $\beta \in \text{Dom}(p_\zeta)$ ,  $[f_\beta(i) = j] \in q_\sigma$ ,  $[f_\beta(i_{\zeta,\beta}) = \delta(\sigma)] \in p_\zeta(\beta)$  and the contradiction is about  $jR_0\delta(\sigma)$ . So  $\beta \in \text{Dom}(p_\zeta)$  (by the last sentence), and  $j \in \text{dom}(p_\zeta(0))$  (as  $p_\zeta(0)$  forces a truth value to  $jR_0\delta(\sigma)$ ), hence by (f),  $[j \notin \text{Rang } f_\beta] \in p_\zeta(\beta)$  or  $(\exists i)([f_\beta(i) = j] \in p_\zeta(\beta))$ . In the first case

$$[j \notin \text{Rang } f_\beta] \in p_\zeta(\beta)^{[\delta(\sigma)]} \leq q_\sigma$$

(as  $j < \delta(\sigma)$ ) contradiction, hence the second case occurs. So for some  $i_1 [f_\beta(i_1) = j] \in p_\zeta(\beta)$ , but then  $\kappa_{i_1} < j < \delta(\sigma)$  clearly  $[f_\beta(i_1) = j] \in q_\sigma$ , hence  $i_1 = i$ . So  $[f_\beta(i_{\zeta,\beta}) = \delta(\sigma)]$ ,  $[f_\beta(i) = j]$  belongs to  $p_\zeta(\beta)$  hence, as  $p_\zeta \in D_\alpha$ ,  $p_\zeta^{[\delta(\sigma)]} \uparrow \beta$  force a truth value for  $iR_\beta i_{\zeta,\beta}$  equal to the one  $p_\zeta(0)$  determine for  $jR_0\delta(\sigma)$  and we get an easy contradiction as  $p_\zeta^{[\delta(\sigma)]} \leq q_\sigma$ .

Similarly there is no contradiction between the third and fifth.

What about a contradiction between the second and fifth, i.e., between  $p_\zeta^{[\delta(\theta)]}(0)$  and an instance of (\*\*) which is  $\beta \in \text{Dom}(p_\xi)$ ,  $[f_\beta(i) = j] \in q_\sigma(\beta)$ ,  $[f_\beta(i_{\xi,\beta}) = \delta(\sigma)] \in p_\xi(\beta)$  and the contradiction is about  $jR_0\delta(\sigma)$ ? But in such a case by (d),  $\delta(\sigma) \in \text{dom } p_\zeta(0)$ . As  $p_\alpha(\beta)^{[\delta(\theta)]} \leq q_\sigma(\beta)$  and  $j \in \text{Dom}(p_\zeta(0))$  and  $[f_\beta(i) = j] \in q_\sigma(\beta)$ , necessarily  $j \in \text{dom } p_\xi(0)$ . So both  $p_\zeta$  and  $p_\xi$  force truth

values for  $jR_0\delta(\sigma)$ , but  $p_\zeta \cup p_\xi \in D_\alpha^1$ , hence it is the same and we get a contradiction between the third and fifth, and we finish by the previous case.

Similarly there is no contradiction between the third and fourth.

Next we deal with two instances of the fourth, i.e.,

$$(*) \text{ for } l = 1, 2, \beta_l \in \text{Dom}(p_\zeta), \\ [f_{\beta_l}(i_l) = j_l] \in q_\sigma \text{ and } [f_{\beta_l}(i_{\zeta, \beta_l}) = \delta(\sigma)] \in p_\zeta(\beta_l).$$

Since there is a contradiction between  $j_1 R_0 \delta(\sigma)$ ,  $j_2 R_0 \delta(\sigma)$  it may be assumed that  $j_1 = j_2$  and  $\beta_1 \neq \beta_2$ . But

$$\begin{aligned} \beta_1, \beta_2 &\in \text{Dom}(p_\zeta), \text{ hence by (c)} \\ A'_{\beta_1} \cap A'_{\beta_2} &\subseteq \text{dom}(p_\zeta(0)), \text{ but by the above} \\ \{j_1, \delta(\sigma)\} &\subseteq A'_{\beta_1} \cap A'_{\beta_2} \end{aligned}$$

so  $j_1 R_0 \delta(\sigma) \in p_\zeta(0)$  or  $\neg j_1 R_0 \delta(\sigma) \in p_\zeta(0)$ , so we get a contradiction between the second and one of the instances of  $(*)$  with which we have already dealt.

Similarly there is no contradiction between two instances of the fifth.

Lastly, what about a contradiction between  $(*)$  and  $(**)$ ? So we assume

$$\begin{aligned} \beta_1 &\in \text{Dom}(p_\zeta), [f_{\beta_1}(i_1) = j_1] \in q_\sigma, \\ [f_{\beta_1}(i_{\zeta, \beta_1}) = \delta(\sigma)] &\in p_\zeta(\beta_1); \\ \beta_2 &\in \text{Dom}(p_\xi), [f_{\beta_2}(i_2) = j_2] \in q_\sigma, \\ [f_{\beta_2}(i_{\zeta, \beta_2}) = \delta(\sigma)] &\in p_\xi(\beta_2). \end{aligned}$$

As we can assume that there is a contradiction necessarily  $\beta_1 \neq \beta_2$ ,  $j_1 = j_2 \in A'_{\beta_1} \cap A'_{\beta_2}$ ,  $j_1 < \delta(\sigma)$ , and  $[j_1 \notin \text{Rang } f_{\beta_1}] \notin p_\zeta(\beta_1)$ . Now  $\beta_1 \in \text{Dom}(p_\zeta)$  and  $p_b^{[\delta(\sigma)]} \leq p^{[\delta(\sigma)]} \leq r_\sigma \leq q_\sigma$ , so by the definition of  $p_b$ , and the last sentence, necessarily  $j_1 \in \text{dom}(p_\zeta(0))$ . Similarly,  $j_1 \in \text{dom}(p_\xi(0))$ . Also  $\delta(\sigma) \in \text{dom}(p_\zeta(0))$  (as  $[f_{\beta_1}(i_{\zeta, \beta_1}) = \delta(\sigma)] \in p_\zeta(\beta_1)$  and  $\delta(\sigma) \in \text{dom}(p_\xi(0))$ ). So  $p_\zeta, p_\xi$  determine the truth value of  $j_1 R_0 \delta(\sigma)$ , and in the same way (as  $p_\zeta \cup p_\xi \in D_\alpha^1$ ) we hence reduce the contradiction to a previous case.

*Point (ii).* Why does  $r_\theta(\beta)$  (where  $\beta > 0$ ) satisfy 1.4(C)(d) (one-to-oneness)?

There are three sources of atomic condition  $[f_\beta(i) = j]$  for  $r_\theta(\beta)$ :  $q_\sigma, p_\zeta^{[\delta(\theta)]}, p_\xi^{[\delta(\theta)]}$ . The second and third cannot contradict as  $p_\zeta \cup p_\xi \in D_\alpha^1$ .

Suppose that the first and second contradict. As  $p_\zeta^{[\delta(\sigma)]} \leq q_\sigma$ , the only possibility is that  $[f_\beta(i_1) = \delta(\sigma)] \in p_\zeta$  contradict some member  $[f_\beta(i_2) = j_2]$  of  $q_\sigma$ . As necessarily  $j_2 < \delta(\sigma)$  we conclude  $i_2 = i_1$ .

As  $[f_\beta(i_1) = \delta(\sigma)] \in p_\zeta(\beta)$ , clearly for some  $\xi_0 : \delta(\sigma) = \xi_\beta(\xi_0), \kappa i_1 < \xi_0 < \kappa i_1 + \kappa$ ,

and  $\kappa i_1 < \xi < \xi_0 \Rightarrow [\xi_\beta(\xi) \notin \text{Rang } f_\beta] \in p_\zeta(\beta)$  hence  $\kappa i_1 < \xi < \xi_0 \Rightarrow [\xi_\beta(\xi) \notin \text{Rang } f_\beta] \in r_\sigma(\beta) \subseteq q_\sigma(\beta)$ , contradicting  $j_2 < \delta(\sigma) = \xi_\beta(\xi_0)$ ,  $[f_\beta(i_1) = j_2] \in q_\sigma(\beta)$ .

Similarly the first and third do not contradict.

*Point (iii).* Why does  $r_\theta(\beta)$  (where  $\beta > 0$ ) satisfy 1.4(C)( $\beta$ ) (embedding)?

In the choice of  $q_\sigma$  (in the fact above) and (\*), (\*\*) of the definition of  $r_\theta(0)$  take care of this.

*Point (iv).* Why does  $r_\theta(\beta)$  (where  $\beta > 0$ ) satisfy 1.4(C)( $\gamma$ ) (Rang)?

Left to the reader.

*Point (v).* Why do (A), (B), (C), (D) above hold?

See definition of  $r_0, r_\theta$ .

#### DISCUSSION.

*Question.* Can we get a similar result for two cardinals  $\lambda_1, \lambda_2$  simultaneously (when  $\kappa < \lambda_1 < \lambda_2 < 2^\kappa$ )?

*Question.* Can you classify countable first-order theories by, e.g.,  $T_1 \sim T_2$  iff for any universe of set theory (e.g., which you get by set forcing and cardinal  $\lambda$ ; e.g., such that  $(\exists \kappa)(\aleph_0 < \kappa = \kappa^{<\kappa} < \lambda < 2^\kappa)$ )  $T_1$  has a universal model of power  $\lambda$  iff  $T_2$  has a universal model of power  $\lambda$ ?

*Question.* For which classes, e.g., is there no universal model of power  $\lambda$  if  $(\exists \kappa)(\aleph_0 < \kappa = \kappa^{<\kappa} < \lambda < 2^\kappa)$ , or even if just  $(\exists \mu < \lambda)(2^\mu > \lambda)$ ? (See [9], p. 86 on this.)

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